TORSION OF A CIRCULAR CYLINDRICAL SHAFT WITH A CONICAL PART

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Let us consider a shaft of variable cross-section which has the shape of a circular truncated cone with an angle 2α as one part, and a circular cylinder as the other (see fig.).

It is known that the problem concerning the torsion of a shaft with variable cross-section may be reduced, in accordance with Foppl [1], to determination of a displacement function $\Psi(r, z)$ which on the cross-section of the shaft satisfies the equation

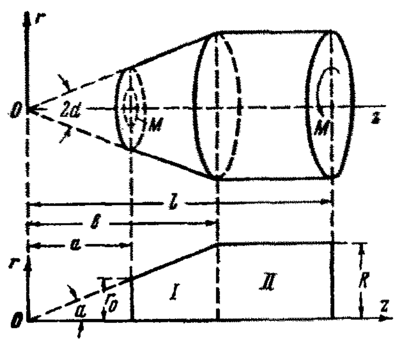
$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{3}{r}\frac{\partial \Psi}{\partial r} + \frac{\partial^2 \Psi}{\partial z^2} = 0 \qquad (1)$$

The stresses $\tau_{\phi r}$, $\tau_{\phi z}$ and the displacement v are determined by $\Psi(r, z)$ through the formulas

$$\tau_{\varphi r} = Gr\frac{\partial \Psi}{\partial r}, \qquad \tau_{\varphi z} = Gr\frac{\partial \Psi}{\partial z} \qquad (2)$$

$$v = r\Psi(r, z) \qquad (3)$$

where G is the shear modulus.



Let the shaft be twisted by means of a torsion moment M_1 applied at the end-face of the conical part, and applied by an arbitrary loading along the side surface and on the end face of the cylindrical part.

The side surface of the conical part of the shaft will be assumed free from applied loads.

Since the side surface of the conical part is free from loads, the projection on this surface of the total stress on the normal to the contour of the axial cross-section of the shaft is equal to zero. This condition may be written down in the form

$$\frac{\tau_{\varphi r}}{\tau_{\varphi z}} = \frac{r}{z} = \tan\alpha \tag{4}$$

The sum of moments of all shear stresses with respect to the axis of the shaft at the end face $c = a$ is equal to the torsion M_1:

$$M_1 = -\int_0^{r_0}\int_0^{2\pi} r^2 \tau_{\varphi z}(r, a)\, dr\, d\varphi \tag{5}$$

The conditions on the side surface and on the end face of the cylindrical part are expressed by the equation

$$\tau_{\varphi r}(R, z) = f_1(z), \qquad \tau_{\varphi z}(r, l) = f_2(r) \tag{6}$$

where the functions $f_1(z)$ and $f_2(r)$ are stepwise continuous and have bounded variation in corresponding intervals.

The solution will be sought in the form

$$\Psi(r, z) = \begin{cases} \Psi_1(r, z) & \text{in region } I, \text{ where } a \leqslant z \leqslant b \\ \Psi_2(r, z) & \text{in region } II, \text{ where } b \leqslant z \leqslant l \end{cases} \tag{7}$$

Along the line of contact of regions I and II we must satisfy the continuity conditions

$$\Psi_1(r, b) = \Psi_2(r, b) \qquad \left(\frac{\partial \Psi_1}{\partial z}\right)_{z=b} = \left(\frac{\partial \Psi_2}{\partial z}\right)_{z=b} \tag{8}$$

Functions $\Psi_1(r, z)$ and $\Psi_2(r, z)$ will be taken in the form

$$\Psi_1(r, z) = \frac{C}{(z^2 + r^2)^{3/2}} \tag{9}$$

$$\Psi_2(r, z) = Az + B(4z^2 - r^2) + D + \frac{1}{r}\sum_{k=1}^{\infty} A_k I_1(\lambda_k r)\cos\lambda_k(z - b) +$$

$$+\frac{1}{r}\sum_{k=1}^{\infty} J_1(s_k r)(B_k \operatorname{sh} s_k z + C_k \operatorname{ch} s_k z) \tag{10}$$

where $\lambda_k = k\pi/l - b$, $s_k = \mu_k/R$, μ_k are the roots of the equation

$$J_2(x) = 0 \tag{11}$$

$J_i(x)$ is a Bessel function of i-th order of the first kind with a real argument, $I_i(x)$ is a Bessel function of the first kind and of imaginary argument [2].

The solution (9) for the conical shaft was given by Foppl [1].

Using relations (2), (9) and (10) we obtain the following expressions.

For the conical part of the shaft

$$\tau_{\varphi z}^{(1)}(r, z) = -3CG\frac{rz}{(z^2+r^2)^{5/2}}, \qquad \tau_{\varphi r}^{(1)}(r, z) = -3CG\frac{r^2}{(z^2+r^2)^{5/2}} \tag{12}$$

For the cylindrical part of the shaft

$$\tau_{\varphi z}^{(2)}(r, z) = G\Big\{Ar + 8Brz - \sum_{k=1}^{\infty} A_k\lambda_k I_1(\lambda_k r)\sin\lambda_k(z-b) +$$

$$+\sum_{k=1}^{\infty}\frac{\mu_k}{R}J_1(s_k r)(B_k \cosh s_k z + C_k \sinh s_k z)\Big\} \tag{13}$$

$$\tau_{\varphi r}^{(2)}(r, z) = G\Big\{-2Br^2 + \sum_{k=1}^{\infty} A_k\lambda_k I_2(\lambda_k r)\cos\lambda_k(z-b) -$$

$$-\sum_{k=1}^{\infty}\frac{\mu_k}{R}J_2(s_k r)(B_k \sinh s_k z + C_k \cosh s_k z)\Big\} \tag{14}$$

It is easy to see that the condition (4) on the side surface of the conical part of the shaft is identically satisfied by expressions (12).

Satisfying condition (5), we obtain

$$C = \frac{M_1}{2G\pi c_0} \quad (-c_0 = \cos^3\alpha - 3\cos\alpha + 2) \tag{15}$$

The second of conditions (6) yields

$$A + 8Bl = \frac{2M_2}{\pi G R^4} \qquad \left(M_2 = 2\pi\int_0^R r^2\tau_{\varphi z}^{(2)}(r, l)\,dr = 2\pi\int_0^R r^2 f_2(r)\,dr\right) \tag{16}$$

Here the value of the following integral was used

$$\int_0^R r^2 J_1(s_k r)\,dr = 0 \tag{17}$$

The first of conditions (6) yields

$$G\left\{-2BR^2 + \sum_{k=1}^{\infty} A_k \lambda_k I_2(\lambda_k R) \cos\lambda_k (z-b)\right\} = f_1(z) \tag{18}$$

Introducing the notations

$$a_0 = \frac{1}{l-b}\int_b^l f_1(z)\,dz, \qquad a_k = \frac{2}{l-b}\int_b^l f_1(z)\cos\lambda_k(z-b)\,dz \tag{19}$$

we obtain from (18)

$$B = -\frac{a_0}{2GR^2}, \qquad A_k = \frac{a_k}{G\lambda_k I_2(\lambda_k R)} \tag{20}$$

Satisfying the first of conditions (8) we obtain

$$\frac{Cr}{(b^2+r^2)^{3/2}} = (Ab + 4Bb^2 + D)\,r - Br^3 + \sum_{k=1}^{\infty} A_k I_1(\lambda_k r) +$$

$$+ \sum_{k=1}^{\infty} J_1(s_k r)(B_k \sinh s_k b + C_k \cosh s_k b) \tag{21}$$

Multiplying (21) by r^2 and integrating with respect to r between the limits zero and R, we obtain

$$Cbd_0 = D\frac{R^4}{4} + Ab\frac{R^4}{4} + BR^4\left(b^2 - \frac{R^2}{6}\right) + R^2\sum_{k=1}^{\infty}\frac{A_k I_2(\lambda_k R)}{\lambda_k} \tag{22}$$

where the following notation has been introduced

$$d_0 = \frac{1}{b}\int_0^R \frac{r^3\,dr}{(b^2+r^2)^{3/2}} = \frac{(1-\cos\alpha)^2}{\cos\alpha} \tag{23}$$

as well as the value of the integral

$$\int_0^R r^2 I_1(\lambda_k r)\,dr = \frac{R^2 I_2(\lambda_k R)}{\lambda_k} \tag{24}$$

Multiplying (21) by $rJ_1(s_p r)$ and integrating the solution obtained with respect to r between the same limits, we obtain

$$C\int_0^R \frac{r^2 J_1(s_p r)\,dr}{(b^2+r^2)^{3/2}} = -2B\,\frac{R^5 J_1(\mu_p)}{\mu_p^2} + R^3 J_1(\mu_p)\sum_{k=1}^{\infty}\frac{A_k\lambda_k I_2(\lambda_k R)}{\mu_p^2 + \lambda_k^2 R^2} +$$

$$+ \frac{R^2 J_1^2(\mu_p)}{2}(B_p \sinh s_p b + C_p \cosh s_p b) \tag{25}$$

For the integrals the following values were used

$$\int_0^R r^4 J_1(s_p r)\,dr = \frac{2R^5}{\mu_p^2} J_1(\mu_p) \tag{26}$$

$$\int_0^R r I_1(\lambda_k r) J_1(s_p r)\,dp = \frac{\lambda_k R^3 I_2(\lambda_k R) J_1(\mu_p)}{\mu_p^2 + \lambda_k^2 R^2}$$

$$\int_0^R r J_1(s_k r) J_1(s_p r)\,dr = \begin{cases} 0 & \text{for } k \neq p \\ {}^1/_2 R^2 J^2{}_1(\mu_p) & \text{for } k = p \end{cases} \tag{27}$$

In an analogous manner, the second of conditions (8) yields

$$A + 8Bb = -\frac{4Cc_0}{R^4} \tag{28}$$

$$3Cb \int_0^R \frac{r^2 J_1(s_p r)\,dr}{(b^2 + r^2)^{5/2}} = \frac{\mu_p R J_1^2(\mu_p)}{2} (B_p \cosh s_p b + C_p \sinh s_p b) \tag{29}$$

where the expressions (17), (15) and (28) were used.

From relations (15), (16), (20) and (28) it follows that

$$\frac{M_2 + M_1}{2\pi (l - b) R^2} + a_0 = 0 \tag{30}$$

which represents the equilibrium equation relating forces which twist the bar.

From relations (22) and (16), taking into account (20) and (15), we obtain

$$A = -2\,\frac{M_1 l + M_2 b}{\pi G (l - b) R^4} \tag{31}$$

$$D = \frac{M_2 (b^2 + {}^1/_6 R^2) + M_1 (2lb - b^2 + {}^1/_6 R^2)}{\pi G (l - b) R^4} - \frac{4}{GR^2} \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} + \frac{2M_1 d_0 b}{\pi G c_0 R^4} \tag{32}$$

From (25) and (29) we find

$$C_p = N_p \cosh s_p b - M_p \sinh s_p b, \quad B_p = M_p \cosh s_p b - N_p \sinh s_p b \tag{33}$$

where

$$M_p = -\frac{-3bM_1}{\pi G c_0 \mu_p R J_1^2(\mu_p)} \int_0^R \frac{r^2 J_1(s_p r)\,dr}{(b^2 + r^2)^{5/2}} \tag{34}$$

$$N_p = \frac{M_1}{\pi G c_0 R^2 J_1^2(\mu_p)} \int_0^R \frac{r^2 J_1(s_p r)\,dr}{(b^2 + r^2)^{3/2}} +$$

$$+ \frac{M_2 + M_1}{\pi G(l - b) R \mu_p^2 J_1(\mu_p)} - \frac{2R}{G J_1(\mu_p)} \sum_{k=1}^{\infty} \frac{a_k}{\mu_p^2 + \lambda_k^2 R^2} \tag{35}$$

Substituting the found values of the coefficients into the formulas (2) and (3), we may determine the stresses $\tau_{\phi r}$, $\tau_{\phi z}$ and the displacement v.

The integrals which enter into expressions (34) and (35) may be evaluated, if they are represented in the form of a series. For example, for the integral

$$\int \frac{x^2 J_1(sx)\,dx}{(x^2 + b^2)^{3/2}}$$

we obtain, integrating by parts,

$$\int \frac{x^2 J_1(xs)\,dx}{(b^2 + x^2)^{3/2}} = - \frac{x J_1(sx)}{(x^2 + b^2)^{1/2}} + s \int \frac{x J_0(sx)\,dx}{(x^2 + b^2)^{1/2}} =$$

$$= - \frac{x J_1(sx)}{(x^2 + b^2)^{1/2}} + s(x^2 + b^2)^{1/2} J_0(sx) + s^2 \int (x^2 + b^2)^{1/2} J_1(sx)\,dx =$$

$$= \sum_{k=0}^{\infty} \frac{s^k (x^2 + b^2)^{k - 1/2} (2k + 1)}{(2k + 1)!!} \frac{J_{k-1}(sx)}{x^{k-1}} \tag{36}$$

In an analogous manner

$$\int \frac{x^2 J_1(sx)\,dx}{(b^2 + x^2)^{5/2}} = - \frac{x J_1(xs)}{3(x^2 + b^2)^{3/2}} - \frac{1}{3} \sum_{k=0}^{\infty} \frac{s^{k+1} (x^2 + b^2)^{k - 1/2} (2k \mp 1)}{(2k + 1)!!} \frac{J_k(sx)}{x^k} \tag{37}$$

In the particular case where the side surface is free from traction we obtain

$$a_0 = a_k = 0, \qquad M_1 = - M_2 \tag{38}$$

The solution for the cylindrical part of the shaft takes the form

$$\Psi_2(r, z) = Az + D + \frac{1}{r} \sum_{k=1}^{\infty} J_1(s_k r)(B_k \sinh s_k z + C_k \cosh s_k z) \tag{39}$$

where

$$A = -\frac{2M_1}{\pi G R^4}, \qquad D = \frac{2M_1 b}{\pi G R^4}\left(1 + \frac{d_0}{c_0}\right) \tag{40}$$

$$C_p = \frac{M_1}{\pi G c_0 \mu_p R J_1^2(\mu_p)} \left[s_p \cosh s_p b \int^{R} \frac{r^2 J_1(s_p r)\, dr}{(b^2 + r^2)^{3/2}} + 3b \sinh s_p b \int_0^R \frac{r^2 J_1(s_p r)\, dr}{(b^2 + r^2)^{5/2}} \right] \tag{41}$$

$$B_p = -\frac{M_1}{\pi G c_0 \mu R J_1^2(\mu_p)} \left[s_p \sinh s_p b \int_0^R \frac{r^2 J_1(s_p r)\, dr}{(b^2 + r^2)^{3/2}} + 3b \cosh s_p b \int_0^R \frac{r^2 J_1(s_p r)\, dr}{(b^2 + r^2)^{5/2}} \right]$$

BIBLIOGRAPHY

1. Föppl, A., Über die Torsion von runden Stäben mit veränderlichen Durchmessern. *Sitzungsberichte Bayerischer Akademie der Wissenschaften*. Vol. 35, Correction 504, pp. 249-262, München, 1905.

2. Gray, E. and Matthews, G.B., *Bessel Functions and their Applications in Physics and Mechanics*. Gosinoizdat, Moscow, 1953.

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